Random graphs and simplicial complexes

Finding the thresholds for existence of important graph properties in random models of graphs and complexes.

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Abstract

Consider an internet website. It has several pages and it might have some links to move from one page to another. How can one determine if it is possible to move from any page to any other page by a sequence of links? How can one determine how many links should one use to do so? Is there a way to build my website in a way that minimizes this number of links?

The answers to this question are just few of the known applications of graph theory. Graph theory is useful not only in web constructing but also in the fields of search engines, data structures, and even in the solution to the problem of map coloring.

Graph theory is, as expected, the branch of mathematics dedicated to the study of graphs. A graph is a mathematical structure, consisting of a set of vertices along with a set of edges, each connecting two vertices of the graph, just as in Figure 1.

[Image: An example of a graph created in the random process of \( G(n,p) \)]

In the example presented in the first paragraph, we can identify the pages of the website with vertices of a graph, and have an edge between two of them if there is a link going from one to the other. This will convert all of the questions above to standard, well-studied, questions in graph theory.

An important and growing part of graph theory is random graph theory. There is a famous model (and some other less famous models) of constructing a graph using some random process – taking \( n \) vertices and for any possible edge, include it in the graph in probability \( p \), and exclude in probability \( 1-p \). In this model an interesting phenomenon occurs: for a given \( p \), certain properties of a graph exist in probability going either to 1 or to 0 (when we take bigger and bigger \( n \)). Finding the threshold for \( p \) for which some graph property
almost surely exists is a very important and vastly studied problem in mathematics, with many applications in computer science.

In our research, we are interested in finding the threshold for the existence of some interesting graph properties, and trying to generalize these results to the multi-dimensional version of graphs – simplicial complexes. We will generalize the random graph model to a random model for simplicial complexes, and find, through mathematical tools and possibly coding and computer experiments, the correct analogs for the results we had about graphs.

**Student mission / Objective:**
Finding the threshold functions $p(n)$ for connectivity, phase transition and other important graph properties in the random graph model. Define a random model for simplicial complexes and write a code line to generate random simplicial complexes. Extend the results from graphs to complexes.

**Requirements:**
Inclination towards logical and abstract thinking and sometimes using drawings as intuition.
It should be mentioned that this is a theoretical project and therefore almost all work will be done using pencil and paper, and sometimes computers. So willingness to work on that kind of a project is needed.
Basic coding knowledge (at least one basic programming language, no need to be an expert programmer)

**Please read the following paper in order to get a sense of the project**
I have attached a paper ("Random Graphs" [2010]) walking through some of the basic known results in the famous random graph model. One can also use the following link to the paper - http://www.cs.cornell.edu/courses/cs4850/2010sp/Course%20Notes/Random-graphs-from-jeh-Feb-06-2010.pdf

**Questions about the paper**
1. What is the motivation to the threshold of connectivity and the emerging of cycles? How are those related to each other?
2. Which graph properties has this threshold phenomenon and which do not?

We will discuss the answers when we meet at the dinner in the opening ceremony.

Please feel free to contact me with questions regarding the project at aameer2@gmail.com

**Recommended reading material**
I have attached an additional paper by Erdos and Renyi from 1959 concerning a similar random process of graphs.